ON THE INTEGRAL MEANS AND SCHWARZIAN DERIVATIVE

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ABSTRACT. Let f be locally univalent in the unit disc $\mathbf{D}$. We study the integral means of powers $|f'|^p$ and obtain upper bounds for them in terms of a solution of an algebraic equation of degree 4. The coefficients of this algebraic equation depend on the parameter p and on the hyperbolic norms of the pre-Schwarzian f''/f' and Schwarzian derivative Sf. We present some results derived from numerical analysis of the polynomial equation.

1. INTRODUCTION

Let f be bounded and univalent in the unit disc $\mathbf{D}$. For $0 \leq r < 1$ and $-\infty < p < \infty$ we consider the integral means

$$\mathrm{I}_p(f, r) = \int_0^{2\pi} |f'(re^{it})|^p dt$$

and define

$$\beta_f(p) = \limsup_{r \to 1} \frac{\log \mathrm{I}_p(f, r)}{|\log(1 - r)|}.$$

Thus $\beta_f(p)$ is the smallest number α such that for each $\epsilon > 0$

$$\mathrm{I}_p(f, r) = \mathrm{O}((1 - r)^{-\alpha - \epsilon}) \ , \ r \to 1 \,.$$

The *universal means spectrum* is defined by

$$\mathrm{B}(p) = \sup\{ \beta_f(p) \ : \ f \text{ bounded univalent} \} \,.$$

It is not difficult to see that $\beta_f(p)$ is convex and

$$\beta_f(p \pm q) \leq \beta_f(p) + q \ , \ q \geq 0 \,.$$

Both properties are inherited by $B(p)$ and become useful tools in estimating $\mathrm{B}(p)$ once it is known at specific values. The function $\mathrm{B}(p)$ has been determined for certain ranges of p and its full determination

would solve several interesting problems in geometric function theory. For example, it has been established that

$$\mathrm{B}(p) = p - 1 \ , \ p \geq 2 \, ,$$

and that

$$\mathrm{B}(p) = |p| - 1 \ , \ p \leq q_0$$

for some $q_0 \leq -2$, [5], [15]. Brennan had earlier conjectured that $\mathrm{B}(p) = |p| - 1$ for $p \leq -2$, [3]. As it turns out, this would follow from just showing that $\mathrm{B}(-2) = 1$.

Among positive values of p the case $p = 1$ has particular interest because it is connected with the coefficient problem. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \, .$$

Then

$$n|a_n| \leq \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} |f'(re^{it})| dt \, , \tag{1.1}$$

and choosing $r = 1 - 1/n$ it follows that for each $\epsilon > 0$

$$n|a_n| = \mathrm{O}(n^{\beta_f(1)+\epsilon}) = \mathrm{O}(n^{\mathrm{B}(1)+\epsilon}) \ \ , \ n \to \infty \, .$$

A remarkable result of Carleson and Jones [4] shows that one did not lose much in taking the absolute value inside the integral in (1.1). They proved that if γ is defined by

$$\gamma = \sup\{\limsup_{n\to\infty} \frac{\log(n|a_n|)}{\log n} : f \text{ bounded univalent}\}$$

then

$$\gamma = \mathrm{B}(1) \, .$$

By convexity $\mathrm{B}(1) \leq 0.5$, and much work has been devoted to finding its exact value. For example, it has been shown by analytic means that

$$0.17 < \mathrm{B}(1) < 0.4886 \, ,$$

see [9], [12], [13, Sect. 5.2], [14, Sect. 8.4]. Numerical experimentation carried out by Carleson and Jones suggested that $\mathrm{B}(1) > 0.23$. This led them to conjecture that $\mathrm{B}(1) = 1/4$ [4].

On the other hand, using a different experimental approach, P. Kraetzer was led in [10], [11] to results that motivated the conjecture that

$$\mathrm{B}(p) = \frac{p^2}{4} \ , \ |p| \leq 2 \, . \tag{1.2}$$

If true, it would solve the Brennan conjecture by setting $p = -2$, and the Carleson-Jones conjecture for $p = 1$.

In this paper we shall show how to estimate $\beta_f(p)$ in terms of the (hyperbolic) norm of the quantities f''/f' and the Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$. This, on the one hand, will allow us to estimate integral means for functions in the *Nehari class*, and on the other, the estimate will be applicable to functions not necessarily univalent. The bound that we obtain for $\beta_f(p)$ is a solution of an algebraic equation of degree 4, and thus can be estimated numerically. We present these results in the last section.

2. RESULTS

Our main result, stated below, only requires the function f to be locally univalent in $\mathbf{D}$.

Theorem 1 : *Suppose that*

$$(1 - |z|)|\frac{f''(z)}{f'(z)}| \le a \quad , \ r_0 \le |z| < 1 , \tag{2.1}$$

and

$$(1 - |z|)^2 |Sf(z)| \le b \quad , \ r_0 \le |z| < 1 . \tag{2.2}$$

Then

$$\beta_f(p) \le \beta$$

where β is the smallest positive solution of

$$\beta(\beta+1)(\beta+2)(\beta+3) - (4|p+1|b + (p+1)^2 a^2)\beta(\beta+1) = 4p^2b^2 . \tag{2.3}$$

Proof : Let

$$g(z) = (f'(z))^{p/2} = \sum_{n=0}^{\infty} b_n z^n \tag{2.4}$$

and consider

$$\mathrm{I}(r) = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{it})|^p dt = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{it})|^2 dt .$$

Then

$$\mathrm{I}(r) = \sum_{n=0}^{\infty} |b_n|^2 r^{2n} ,$$

$$\mathrm{I}''(r) = \sum_{n=1}^{\infty} 2n(2n - 1)|b_n|^2 r^{2n-2} ,$$

$$\mathrm{I}''''(r) = \sum_{n=2}^{\infty} 2n(2n - 1)(2n - 2)(2n - 3)|b_n|^2 r^{2n-4}$$

$$\leq \sum_{n=2}^{\infty} [2n(2n-2)]^2 |b_n|^2 r^{2n-4} = 16 \sum_{n=2}^{\infty} [n(n-1)]^2 |b_n|^2 r^{2n-4} \,.$$

It follows that

$$\mathrm{I}''''(r) \leq \frac{16}{2\pi} \int_0^{2\pi} |g''(re^{it})|^2 dt \,. \tag{2.5}$$

From (2.4) we have

$$g' = \frac{p}{2}(f')^{\frac{p}{2}} \frac{f''}{f'}$$

and

$$g'' = \frac{p}{2}(f')^{\frac{p}{2}} \left(\frac{f'''}{f'} + \frac{p-2}{2}(\frac{f''}{f'})^2 \right)$$

$$= \frac{p}{2}(f')^{\frac{p}{2}} \left(Sf + \frac{p+1}{2}(\frac{f''}{f'})^2 \right) \,.$$

This inserted in (2.5) gives

$$\mathrm{I}''''(r) \leq \frac{4p^2}{2\pi} \int_0^{2\pi} |f'|^p \left(|Sf|^2 + |p+1| \frac{f''}{f'}|^2 |Sf| + \frac{(p+1)^2}{4} |\frac{f''}{f'}|^4 \right) dt \,,$$

where the integrand is evaluated at re^{it}. Thus for $r \geq r_0$ we obtain

$$\mathrm{I}''''(r) \leq \frac{4p^2}{2\pi} \int_0^{2\pi} |f'|^p \left(\frac{b^2}{(1-r)^4} + \frac{4|p+1|b + (p+1)^2 a^2}{4(1-r)^2} |\frac{f''}{f'}|^2 \right) dt \,.$$

Also

$$\mathrm{I}''(r) + \frac{\mathrm{I}'(r)}{r} = \sum_{n=1}^{\infty} 4n^2 |b_n|^2 r^{2n-2}$$

$$= \frac{4}{2\pi} \int_0^{2\pi} |g'(re^{it})|^2 dt = \frac{p^2}{2\pi} \int_0^{2\pi} |f'|^p |\frac{f''}{f'}|^2 dt \,.$$

Hence for each $\delta > 0$ and r sufficiently close to 1

$$\mathrm{I}''''(r) \leq \frac{4p^2 b^2}{(1-r)^4} \mathrm{I}(r) + \frac{4|p+1|b + (p+1)^2 a^2 + \delta}{(1-r)^2} \mathrm{I}''(r) \tag{2.6}$$

because $\mathrm{I}'(r) = o(\mathrm{I}''(r))$ as $r \to 1$.

The differential equation

$$u''''(r) = \frac{4p^2 b^2}{(1-r)^4} u(r) + \frac{4|p+1|b + (p+1)^2 a^2 + \delta}{(1-r)^2} u''(r)$$

admits the solution

$$u(r) = \frac{B}{(1-r)^\beta}$$

where

$$\beta(\beta+1)(\beta+2)(\beta+3) - (4|p+1|b + (p+1)^2 a^2 + \delta)\beta(\beta+1) = 4p^2 b^2 \,.$$

Therefore if $B > 0$ is chosen sufficiently large and r is close to 1 then

$$\mathrm{I}(r) \leq u(r) \,.$$

By letting $\delta \to 0$ we obtain the conclusion of the theorem.

The Nehari class is the set N of all (univalent) functions defined in **D** such that

$$(1 - |z|^2)^2 |Sf(z)| \leq 2 \,.$$

In [6] it was shown that any such function can be normalized so that $f''(0) = 0$ while still remaning analytic. This gives rise to the class N_0. The normalization is accomplished by considering a suitable Möbius shift from the left, which as mentioned above does not introduce a pole. Also, when normalized, the image $f(\mathbf{D})$ will either be bounded or else a parallel strip.

Theorem 2: *Suppose that $f''(0) = 0$ and*

$$(1 - |z|^2)^2 |Sf(z)| \leq 2s \leq 2 \,. \tag{2.7}$$

Then for $p \geq -1$

$$\beta_f(p) \leq \beta$$

where β is the unique positive solution of

$$\beta(\beta+1)[(\beta+2)(\beta+3) - (p+1)(2s + (p+1)(1 - \sqrt{1-s})^2)] = p^2 s^2 \,. \tag{2.8}$$

Proof: It order to apply Theorem 1 we need to establish the bound for $y = f''/f'$. Since

$$y' = \frac{1}{2}y^2 + Sf$$

it is not difficult to see that $|y(z)| \leq v(|z|)$ where $v = v(x)$ is the solution to the first order equation

$$v' = \frac{1}{2}v^2 + \frac{2s}{(1 - x^2)^2} \quad , \quad v(0) = 0 \,.$$

For details, see [7]. This leads to the estimate

$$\limsup_{|z| \to 1}(1 - |z|^2)|\frac{f''(z)}{f'(z)}| \leq \limsup_{x \to 1}(1 - x^2)v(x) = 2(1 - \sqrt{1-s}) \,.$$

It follows that for any number $a > 1 - \sqrt{1-s}$ there exists $0 \leq r_0 < 1$ such that

$$(1 - |z|)|\frac{f''(z)}{f'(z)}| \leq a \quad , \quad r_0 \leq |z| < 1 \,.$$

Thus we can use Theorem 1 for any $a > 1 - \sqrt{1-s}$, $b > s/2$ and suitable r_0. In the limiting case we obtain the conclusion of the theorem.

Since Theorem 1 only requires estimates near the boundary it is natural to impose restrictions only on the upper limits of the pre-Schwarzian and Schwarzian norms. In this case we can also drop the normalization $f''(0) = 0$.

Theorem 3: *Suppose that f is bounded and*

$$\limsup_{|z|\to 1}(1 - |z|^2)^2|Sf(z)| \le 2s < 2\,.$$

Then for $p \ge -1$

$$\beta_f(p) \le \beta$$

where β is the unique positive solution of

$$\beta(\beta + 1)[(\beta + 2)(\beta + 3) - (p + 1)(2s + (p + 1)(1 - \sqrt{1 - s})^2)] = p^2 s^2\,.$$

Remark: Note that functions satisfying the hypothesis of the theorem are not necessarily univalent.

Proof: We need to estimate $(1 - |z|)|f''/f'|$ near the boundary. Let $y = f''/f'$ and let s' be such that $2s < 2s' < 2$. Choose r_0 so that

$$(1 - |z|^2)^2|Sf(z)| \le 2s'$$

for $r_0 \le |z| < 1$. Since

$$y' = \frac{1}{2}y^2 + Sf$$

we shall use a comparison argument as before to bound $|y(z)|$ in terms of the solution $v = v(x)$ of

$$v' = \frac{1}{2}v^2 + \frac{2s'}{(1 - x^2)^2}\quad,\ v(0) = 0\,.$$

Let ζ be fixed with $|\zeta| = 1$. If $|y(r_0\zeta)| \le v(r_0)$ then

$$|y(x\zeta)| \le v(x)\quad,r_0 \le x < 1\,.$$

Thus

$$\limsup_{x\to 1}(1 - x^2)|y(x\zeta)| \le \limsup_{x\to 1}(1 - x^2)v(x) = 2(1 - \sqrt{1 - s'})\,. \quad (2.9)$$

If $|y(r_0\zeta)| > v(r_0)$ we need to consider a Möbius transformation $g = T \circ f$ so that $w = g''/g'$ has an appropriate initial condition at $z = r_0\zeta$. Let

$$g = \frac{f}{1 - cf}\,.$$

Then

$$\frac{g''}{g'} = \frac{f''}{f'} + \frac{2cf'}{1 - cf}\,. \quad (2.10)$$

First choose c so that

$$\left|\frac{f''(r_0\zeta)}{f'(r_0\zeta)} + \frac{2cf'(r_0\zeta)}{1 - cf(r_0\zeta)}\right| < v(r_0)\,. \tag{2.11}$$

This represents an open disc in the variable c. With this, the function $w = g''/g'$ satisfies

$$w' = \frac{1}{2}w^2 + Sf$$

and $|w(r_0\zeta)| < v(r_0\zeta)$. It follows that

$$|w(x\zeta)| \le v(x) \quad, \ r_0 \le x < 1\,. \tag{2.12}$$

In particular, g cannot have a pole in $[r_0\zeta, \zeta)$, which means that f did not assume the value $1/c$ in that segment. It is clear that the restriction (2.11) allows one to choose c so that the stronger condition holds:

$$\frac{1}{c} \notin \overline{f[r_0\zeta, \zeta)}\,.$$

From (2.11) and (2.12) we obtain for $r_0 \le x < 1$

$$|\frac{f''(x\zeta)}{f'(x\zeta)}| \le v(x) + \frac{2|c||f'(x\zeta)|}{|1 - cf(x\zeta)|}\,,$$

hence

$$(1 - x^2)|\frac{f''(x\zeta)}{f'(x\zeta)}| \le (1 - x^2)v(x) + \frac{2|c|(1 - x^2)|f'(x\zeta)|}{|1 - cf(x\zeta)|}\,.$$

Since $(1 - x^2)|f'(x\zeta)| \le 4\text{dist}(f(x\zeta), \partial\Omega)$, $\Omega = f(D)$, and since the term $|1 - cf(x\zeta)|$ remains bounded away from 0, we conclude that, given s'' with $s' < s'' < 1$, there exists $r_1 = r_1(\zeta)$ such that

$$(1 - x^2)|\frac{f''(x\zeta)}{f'(x\zeta)}| \le 2(1 - \sqrt{1 - s''}) \quad, \ r_1 \le x < 1\,.$$

Finally, observe that the parameters in (2.11) depend continuously on ζ, therefore we can make the selection of $c = c(\zeta)$ in a continuous fashion. This implies that r_1 also varies continuously, hence we can make the choice of r_1 independent of ζ. This together with the initial estimate (2.9) implies that for any $s < s'' < 1$

$$\limsup_{|z|\to 1} |\frac{f''(z)}{f'(z)}| \le 2(1 - \sqrt{1 - s''})\,.$$

Hence we can apply Theorem 1 for any $a > 1 - \sqrt{1 - s}$ and $b > s/2$, from which the conclusion of Theorem 3 follows.

3. CONSEQUENCES OF THEOREM 2

In this section we would like to present some consequences and numerical results that can be obtained from Theorem 2.

(i) First let us consider equation (2.8) for small values of s. Then we have
$$\beta(\beta + 1)[(\beta + 2)(\beta + 3) + \mathrm{O}(s)] = p^2 s^2 \quad, \ s \to 0.$$
Since $\beta = 0$ when $s = 0$, it follows by continuity that for small values of s the dominant term in the left hand side is 6β. From this it is not difficult to see that $6\beta = p^2 s^2 + \mathrm{O}(s^3)$. Hence for $p \geq -1$

$$\beta_f(p) \leq \frac{1}{6} p^2 s^2 + \mathrm{O}(s^3) \quad, \ s \to 0. \tag{3.1}$$

It was shown in [8] that there exists a constant $c > 0$ and $s_0 > 0$ small enough such that for each $0 \leq s \leq s_0$ there is a function f satisfying (2.7) for which
$$\beta_f(1) \geq ck^2.$$
Hence (3.1) for $p = 1$ is best possible in order of magnitude.

(ii) Let now $s = 1$ in (2.8). This means, we are considering the full class N_0. We have
$$\beta(\beta + 1)[(\beta + 2)(\beta + 3) - (p + 1)(p + 3)] = p^2,$$
that is
$$\beta^4 + 6\beta^3 + (8 - 4p - p^2)\beta^2 + (3 - 4p - p^2)\beta = p^2. \tag{3.2}$$
This implies that
$$\beta \sim \frac{p^2}{3} \quad, \ p \to 0,$$
hence
$$\beta_f(p) \leq \frac{p^2}{3} + \mathrm{O}(p^3) \quad, \ p \to 0.$$
From (3.2) we can also obtain the following estimates for $\beta_f(p)$, $f \in N_0$:

$p =$	-1	-0.8	-0.6	-0.4	-0.2	-0.1
$\beta_f(p) <$	0.1323	0.0965	0.0632	0.0336	0.0104	0.0030

$p =$	0.1	0.2	0.4	0.6	0.8	1
$\beta_f(p) <$	0.0030	0.0175	0.0874	0.2171	0.3853	0.5726.

Note that, for $p \leq -0.2$, these bounds for N_0 are better than the conjectured bound (1.2) for general univalent functions. Also, since

$\beta_f(p)$ is convex, we can use the estimates for small values of p and the fact that $\beta_f(2) \leq 1$ to obtain the improved estimate

$$\beta_f(1) \leq 0.42\,.$$

Unfortunately this is still far from the bound $\beta_f(1) \leq 0.25$ conjectured by Carleson and Jones. In any case, this should be compared with the bound $B(1) < 0.4886$.

(iii) Suppose that f is univalent in $\mathbf{D}$ and has a k-quasiconformal extension to $\mathbf{C}$, where $0 \leq k \leq 1/3$. Then (2.7) holds with $s = 3k$, hence by (3.1)

$$\beta_f(p) \leq \frac{3}{2}p^2k^2 + \mathrm{O}(k^3) \quad,\ k \to 0\,. \tag{3.3}$$

If p is determined such that $\beta_f(p) = p - 1$ then [14, Cor. 10.18] shows that the Hausdorff dimension satisfies $\dim\partial f(\mathbf{D}) \leq p$. Using (3.3) we obtain first that $p = 1 + \beta_f(p) < 1 + \mathrm{O}(k^2)$, which inserted back into (3.3) gives

$$\dim\partial f(\mathbf{D}) \leq 1 + \frac{3}{2}k^2 + +\mathrm{O}(k^3) \quad,\ k \to 0\,. \tag{3.4}$$

This improves the estimate $1 + 37k^2$ established in [1].

Binder and Rhode [2] have conjectured that (3.3) can be replaced by $\beta_f(p) \leq p^2k^2/4$; compare (1.2). Note that our estimate (3.1) is better provided we replace the assumption of k-quasiconformal extensibility by the stronger assumption (2.7) with $s = k$.

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